How definitive *is* the standard interpretation of Gödel's Incompleteness Theorem?

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Standard interpretations of Gödel's “undecidable” proposition, [(Ax)R(x)], argue that, although [~(Ax)R(x)] is PA-provable if [(Ax)R(x)] is PA-provable, we may not conclude from this that [~(Ax)R(x)] is PA-provable. We show that such interpretations are inconsistent with a standard Deduction Theorem of first order theories.

Contents

1. Introduction

In his seminal 1931 paper [Go31a], Gödel meta-mathematically argues that his “undecidable” proposition, [(Ax)R(x)][2], is such that (cf. [An02b], §1.6(*iv*)):

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> If $[(\mathrm{A}x)R(x)]$ is PA-provable, then $[\sim(\mathrm{A}x)R(x)]$ is PA-provable.

Now, a standard Deduction Theorem of an arbitrary first order theory states that ([Me64], p61, Corollary 2.6):

> If T is a set of well-formed formulas of an arbitrary first order theory K, and if $[A]$ is a closed well-formed formula of K, and if $(T, [A])|\!-_{\mathrm{K}} [B]$, then $T|\!-_{\mathrm{K}} ([A \Rightarrow B])$.

In an earlier essay ([An02b], Appendix 1), we implicitly assumed, without proof, that:

> $(T, [A])|\!-_{\mathrm{K}} [B]$ holds if, and only if, $T|\!-_{\mathrm{K}} [B]$ holds when we assume $T|\!-_{\mathrm{K}} [A]$. (*)

In other words, we assumed that $[B]$ is a deduction from $(T, [A])$ in K if, and only if, whenever $[A]$[3] is a hypothetical deduction from T in K, $[B]$ is a deduction from T in K.

We then argued, that it should follow (essentially by the reasoning in §2.2 below), that:

> $[(\mathrm{A}x)R(x) \Rightarrow \sim(\mathrm{A}x)R(x)]$ is PA-provable,

and, therefore, that:

> $[\sim(\mathrm{A}x)R(x)]$ is PA-provable.

We then concluded that PA is omega-inconsistent.

However, this conclusion is inconsistent with standard interpretations of Gödel's reasoning, which, firstly, assert both $[(\mathrm{A}x)R(x)]$ and $[\sim(\mathrm{A}x)R(x)]$ as PA-unprovable, and, secondly, assume that PA can be omega-consistent. Such interpretations, therefore,

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[2] We use square brackets to differentiate between a formal expression $[F]$ and its interpretation "*F*", where we follow Mendelson's definition of an interpretation M of a formal theory K, and of the interpretation of a formula of K under M ([Me64], p49, §2).

[3] For the purposes of this essay, we assume everywhere that $[A]$ is a closed well-formed formula of K.

implicitly deny that the PA-provability of [~(A*x*)*R*(*x*)] can be inferred from the above meta-argument; ipso facto, they imply that (*) is false.

In the following sections, we review the Deduction Theorems used in the earlier argument, and give a meta-mathematical proof of (*). It follows that the standard interpretations of Gödel's reasoning are inconsistent with a standard Deduction Theorem of an arbitrary first order theory ([Me64], p61, Corollary 2.6). We conclude that such interpretations cannot be accepted as definitive.

1.1 An overview

We first review, in Theorem 1, the proof of a standard Deduction Theorem, if (*T*, [*A*])$|-_K$ [*B*], then *T*$|-_K$ [*A* => *B*], where an explicit deduction of [*B*] from (*T*, [*A*]) is known.

We then show, in Corollary 1.2, that Theorem 1 can be constructively extended to cases where (*T*, [*A*])$|-_K$ [*B*] is established meta-mathematically, and where an explicit deduction of [*B*] from (*T*, [*A*]) is not known.

We finally prove (*) in Theorem 2.

2. A standard Deduction Theorem

The following is, essentially, Mendelson's proof of a standard Deduction Theorem ([Me64], p61, Proposition 2.4) of an arbitrary first order theory K:

Theorem 1: If *T* is a set of well-formed formulas of an arbitrary first order theory K, and if [*A*] is a closed well-formed formula of K, and if (*T*, [*A*])$|-_K$ [*B*], then *T*$|-_K$ [*A* => *B*].

Proof: Let <[B_1], [B_2], ..., [B_n]> be a deduction of [B] from (T, [A]) in K.

Then, by definition, [B_n] is [B] and, for each i, either [B_i] is an axiom of K, or [B_i] is in T, or [B_i] is [A], or [B_i] is a direct consequence by some rules of inference of K of some of the preceding well-formed formulas in the sequence.

We now show, by induction, that T|-$_{\mathrm{K}}$ [A => B_i] for each i =< n. As inductive hypothesis, we assume that the proposition is true for all deductions of length less than n.

(*i*) If [B_i] is an axiom, or belongs to T, then T|-$_{\mathrm{K}}$ [A => B_i], since [B_i => (A => B_i)] is an axiom of K.

(*ii*) If [B_i] is [A], then T|-$_{\mathrm{K}}$ [A => B_i], since T|-$_{\mathrm{K}}$ [A => A].

(*iii*) If there exist j, k less than i such that [B_k] is [B_j => B_i], then, by the inductive hypothesis, T|-$_{\mathrm{K}}$ [A => B_j], and T|-$_{\mathrm{K}}$ [A => (B_j => B_i)]. Hence, T|-$_{\mathrm{K}}$ [A => B_i].

(*iv*) Finally, suppose there is some j < i such that [B_i] is [(Ax)B_i], where x is a variable in K. By hypothesis, T|-$_{\mathrm{K}}$ [A => B_j]. Since x is not a free variable of [A], we have that [(Ax)(A => B_j) => (A => (Ax)B_j] is PA-provable. Since T|-$_{\mathrm{K}}$ [A => B_j], it follows by Generalisation that T|-$_{\mathrm{K}}$ [(Ax)(A => B_j)], and so T|-$_{\mathrm{K}}$ [A => (Ax)B_j], i.e. T|-$_{\mathrm{K}}$ [A => B_i].

This completes the induction, and Theorem 1 follows as the special case where i = n. ¶[4]

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[4] We use the symbol "¶" as an end-of-proof marker.

2.1 A number-theoretic corollary

Now, Gödel has defined ([Go31a], p22, Definition 45(6)) a primitive recursive number-theoretic relation $x\mathrm{B}_{(\mathrm{K},\, T)}y$ that holds if, and only if, x is the Gödel-number of a deduction from T of the K-formula whose Gödel-number is y.

We thus have:

Corollary 1.1[5]: If the Gödel-number of the well-formed K-formula [B] is b, and that of the well-formed K-formula [A => B] is c, then Theorem 1 holds if, and only if[6]:

$$(\mathrm{E}x)x\mathrm{B}_{(\mathrm{K},\, T,\, [A])}b \Rightarrow (\mathrm{E}z)z\mathrm{B}_{(\mathrm{K},\, T)}c$$

2.2 An extended Deduction Theorem

We next consider the proposition:

Corollary 1.2: If we assume Church's Thesis[7], then Theorem 1 holds even if the premise (T, [A])$|\!-_{\mathrm{K}}$ [B] is established meta-mathematically, and a deduction <[B_1], [B_2], ..., [B_n]> of [B] from (T, [A]) in K is not known explicitly.

Proof: Since Gödel's number-theoretic relation $x\mathrm{B}_{(\mathrm{K},\, T)}y$ is primitive recursive, it follows that, if we assume Church's Thesis - which implies that a number-theoretic relation is decidable if, and only if, it is recursive - we can effectively determine some finite natural

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[5] We note that Corollary 1.1 and Corollary 2.2 may be essentially different number-theoretic assertions, which may not be obviously equivalent; the "obvious" assumption (*), thus, may need a proof.

[6] We note that this symbolically expresses a meta-equivalence in a recursive arithmetic RA, based on a semantic interpretation of the definition of the primitive recursive relation $xB_{(\mathrm{K},\, T)}y$; it is not a K-formula.

[7] Church's Thesis: A number-theoretic function is effectively computable if, and only if, it is recursive ([Me64], p147, footnote). We appeal explicitly to Church's Thesis here to avoid implicitly assuming that every recursive relation is algorithmically decidable (cf. [An02c], §II(7) Corollary 14.3). In Anand ([An02g], §2.5(xii)) we show that, under a constructive interpretation of classical foundational concepts, Church's Thesis is a Theorem; such a premise would not, then, be needed.

number n for which the assertion $n\mathrm{B}_{(\mathrm{K},\, T,\, [A])}b$ holds, where the Gödel-number of the well-formed K-formula $[B]$ is b.

Since n would then, by definition, be the Gödel-number of a deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from $(T, [A])$ in K, we may thus constructively conclude, from the meta-mathematically determined assertion $(T, [A])|\text{-}_{\mathrm{K}}\ [B]$, that some deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from $(T, [A])$ in K can, indeed, be effectively determined.

Theorem 1 follows. ¶

3. An additional deduction theorem

We finally prove (*) as an additional deduction theorem, in an arbitrary first order theory K:

Theorem 2: If K is an arbitrary first order theory, and if $[A]$ is a closed well-formed formula of K, then $(T, [A])|\text{-}_{\mathrm{K}}\ [B]$ if, and only if, $T|\text{-}_{\mathrm{K}}\ [B]$ holds when we assume $T|\text{-}_{\mathrm{K}}\ [A]$.

Proof: Firstly, if there is a deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from $(T, [A])$ in K, and there is a deduction $<[A_1], [A_2], ..., [A_m]>$ of $[A]$ from T in K, then $<[A_1], [A_2], ..., [A_m], [B_1], [B_2], ..., [B_n]>$ is a deduction of $[B]$ from T in K. Hence we have: if $(T, [A])|\text{-}_{\mathrm{K}}\ [B]$, then $T|\text{-}_{\mathrm{K}}\ [B]$ holds when we assume $T|\text{-}_{\mathrm{K}}\ [A]$.

Secondly, if there is a deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from T in K, then we have, trivially, that: if $T|\text{-}_{\mathrm{K}}\ [B]$ holds when we assume $T|\text{-}_{\mathrm{K}}\ [A]$, then $(T, [A])|\text{-}_{\mathrm{K}}\ [B]$.

Lastly, we assume that there is no deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from T in K. If, now, $T|\text{-}_{\mathrm{K'}}\ [B]$ holds when we assume $T|\text{-}_{\mathrm{K'}}\ [A]$ in any consistent extension K' of K, then, if we assume that there is a sequence $<[A_1], [A_2], ..., [A_m]>$ of well-formed K'-formulas such that $[A_m]$ is $[A]$ and, for each $m >= i >= 1$, either $[A_i]$ is an axiom of K', or $[A_i]$ is in T, or $[A_i]$ is a direct consequence by some rules of inference of K' of some of the preceding well-formed formulas in the sequence, then we can show, by induction on the deduction length n, that there is a sequence $<[B_1], [B_2], ..., [B_n]>$ of well-formed K-formulas such that $[B_1]$ is $[A]$[8], $[B_n]$ is $[B]$ and, for each $i > 1$, either $[B_i]$ is an axiom of K, or $[B_i]$ is in T, or $[B_i]$ is a direct consequence by some rules of inference of K of some of the preceding well-formed formulas in the sequence.

Hence, if there is a deduction $<[A_1], [A_2], ..., [A_m]>$ of $[A]$ from T in K', then $<[A_1], [A_2], ..., [A_m], [B_2], ..., [B_n]>$ is a deduction of $[B]$ from T in K'. By definition, it follows that $<[B_2], ..., [B_n]>$ is a deduction of $[B]$ from $(T, [A])$ in K. We thus have: if $T|\!-_{\mathrm{K}} [B]$ holds when we assume $T|\!-_{\mathrm{K}} [A]$, then $(T, [A])|\!-_{\mathrm{K}} [B]$. This completes the proof. ¶

In view of Corollary 1.2, we thus have:

Corollary 2.1: If we assume Church's Thesis, and if $[A]$ is a closed well-formed formula of K, then we may conclude $T|\!-_{\mathrm{K}} ([A] => [B])$ if $T|\!-_{\mathrm{K}} [B]$ holds when we assume $T|\!-_{\mathrm{K}} [A]$.[9]

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[8] $[A]$ is thus the hypothesis in the sequence; it is the only well-formed K-formula in the sequence that is not an axiom of K, not in T, and not a direct consequence of the axioms of K by any rules of inference of K.

[9] We note that there is a model-theoretic proof of Corollary 2.1. The case $T|\!-_{\mathrm{K}} [B]$ is straightforward.

If $T|\!-_{\mathrm{K}} [B]$ does not hold, then, as noted in Theorem 2, if $T|\!-_{\mathrm{K}} [B]$ holds when we assume $T|\!-_{\mathrm{K}} [A]$, then there is a sequence $<[B_1], [B_2], ..., [B_n]>$ of well-formed K-formulas such that $[B_1]$ is $[A]$, $[B_n]$ is $[B]$ and, for each $i > 1$, either $[B_i]$ is an axiom of K, or $[B_i]$ is in T, or $[B_i]$ is a direct consequence by some rules of inference of K of some of the preceding well-formed formulas in the sequence.

(Note: In the following, if T is the set of well-formed K-formulas $\{[T_1], [T_2], ..., [T_l]\}$ then $(T \,\&\, [A])$ denotes the well-formed K-formula $[T_1 \,\&\, T_2 \,\&\, ..., T_l \,\&\, A]$, and, $(T \,\&\, A)$ denotes its interpretation in M: $T_1 \,\&\, T_2 \,\&\, ..., T_l \,\&\, A$.)

If, now, any well-formed formula in $(T, [A])$ is false under an interpretation M of K, then $(T \,\&\, A) => B$ is vacuously true in M.

If, however, all the well-formed formulas in $(T, [A])$ are true under interpretation in M, then the sequence $<[B_1], [B_2], ..., [B_n]>$ interprets as a deduction in M, since the interpretation preserves the axioms and rules of inference of K (cf. [Me64], p57). Thus $[B]$ is true in M, and so is $(T \,\&\, A) => B$.

In other words, we cannot have $(T, [A])$ true and $[B]$ false in M as this would imply that there is some consistent extension K' of K in which $T|\!-_{\mathrm{K'}} [A]$, but not $T|\!-_{\mathrm{K'}} [B]$, which is contrary to the hypothesis that, in any consistent K in which we assume $T|\!-_{\mathrm{K}} [A]$, we also have $T|\!-_{\mathrm{K}} [B]$.

We note that, in the notation of Corollary 1.1, if the Gödel-number of the well-formed K-formula [A] is a, then Corollary 2.1 holds if, and only if[10]:

Corollary 2.2: $((\mathrm{E}x)x\mathrm{B}_{(\mathrm{K},\,T)}a \Rightarrow (\mathrm{E}u)u\mathrm{B}_{(\mathrm{K},\,T)}b) \Rightarrow (\mathrm{E}z)z\mathrm{B}_{(\mathrm{K},\,T)}c$.

4. Conclusion

Since standard interpretations of Gödel's reasoning and conclusions do not admit Theorem 2 as a valid inference, such interpretations are inconsistent with the standard Deduction Theorem for an arbitrary first order theory [Me64], p61, Proposition 2.4); they cannot, therefore, be considered definitive.

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Hence, (T & A) => B is true in all models of K. By a consequence of Gödel's Completeness Theorem for an arbitrary first order theory ([Me64], p68, Corollary 2.15(a)), it follows that $|\!-_{\mathrm{K}}$ (T & [A]) => [B]), and, ipso facto, that $T|\!-_{\mathrm{K}}$ ([A] => [B]).

[10] We note that this, too, is not a K-formula, but a semantic meta-equivalence, based on the definition of the primitive recursive relation $xB_{(\mathrm{K},\,T)}y$.

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